

# Non-split Brauer-Severi varieties do not admit full exceptional collections

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## Abstract

Recently, Novaković conjectured that non-split Brauer-Severi varieties do not admit full strong exceptional collections. In this short note, we explain how a stronger version of this conjecture follows easily from known results on noncommutative motives.

## 1 Introduction

For an arbitrary field  $k$ , Novaković stated the following as a conjecture in [3]:

**Conjecture 1.1.** Let  $X \neq \mathbb{P}_k^n$  be a  $n$ -dimensional Brauer-Severi variety. Then  $D^b(X)$  does not admit a full strongly exceptional collection.

He proves the conjecture in dimension  $n \leq 3$  [4] by exploiting the transitivity of the braid group action on full exceptional collections for  $\mathbb{P}_k^n$  to reduce to an equivalence  $D^b(A) \cong D^b(k)$ . If  $A \cong M_l(D)$ , for  $D$  a division algebra over  $k$ , these are just the categories of  $\mathbb{Z}$ -graded vector spaces over  $D$ , respectively  $k$ , so there is an equivalence only if  $D$  is isomorphic to  $k$ . Since the transitivity of the braid group action (which is only established for  $n \leq 3$ ) is only used to be able to reduce to a single semi-orthogonal component, this suggests that noncommutative motives might provide the right framework for this conjecture. Using some results from [7] on noncommutative motives of separable algebras, we prove a slightly stronger version of Conjecture 1.1, showing that non-split Brauer-Severi varieties do not admit full étale exceptional collections.

## 2 Noncommutative motives of separable algebras

To any small dg-category  $\mathcal{A}$ , one can associate (functorially) its noncommutative motive  $U(\mathcal{A})$ , which takes values in a category  $\mathbf{Hmo}_0(k)$ . This category has as objects small dg-categories, and for two such categories  $\mathcal{A}$  and  $\mathcal{B}$ ,

$$\mathrm{Hom}_{\mathbf{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) \cong K_0\mathbf{rep}(\mathcal{A}, \mathcal{B}),$$

where  $\mathbf{rep}(\mathcal{A}, \mathcal{B})$  is the full triangulated subcategory of  $D(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$  consisting of those  $\mathcal{A}$ - $\mathcal{B}$ -bimodules  $B$  such that for every  $x \in \mathcal{A}$ , the right  $\mathcal{B}$ -module  $B(x, -)$  is a compact object in  $D(\mathcal{B})$ . The composition is induced by the derived tensor product of bimodules.

More details on the construction of  $U$  can be found in [6], but for the purposes of this note, we will only need that  $U$  is a “universal additive invariant”. An additive invariant is any functor  $E : \mathbf{dgc}(\mathcal{A}) \rightarrow D$  taking values in an additive category  $D$  such that:

1. it sends dg-Morita equivalences to isomorphisms,
2. for any pre-triangulated dg-category  $\mathcal{A}$ , with full pre-triangulated dg-subcategories  $\mathcal{B}$  and  $\mathcal{C}$  giving rise to a semi-orthogonal decomposition

$$\mathbf{H}^0(\mathcal{A}) = \langle \mathbf{H}^0(\mathcal{B}), \mathbf{H}^0(\mathcal{C}) \rangle,$$

the morphism  $E(\mathcal{B}) \oplus E(\mathcal{C}) \rightarrow E(\mathcal{A})$  induced by the inclusions is an isomorphism.

We now review some results from [7]. Remember that the category of noncommutative Chow motives  $\mathbf{NChow}(k)$  is defined as the idempotent completion of the full subcategory of  $\mathbf{Hmo}_0(k)$  containing the smooth and proper dg-categories. Now let  $\mathbf{Sep}(k)$  (respectively  $\mathbf{CSep}(k)$ ) denote the full subcategory of  $\mathbf{NChow}(k)$  consisting of the  $U(A)$ , for  $A$  a separable (respectively commutative separable)  $k$ -algebra. Also let  $\mathbf{CSA}(k)^{\oplus}$  denote the closure under finite direct sums of the full subcategory of  $\mathbf{NChow}(k)$  consisting of the  $U(A)$ , for  $A$  a central simple  $k$ -algebras. Note that the  $\oplus$  is there since central simple  $k$ -algebras are not closed under products, whereas (commutative) separable algebras are. In this way  $\mathbf{Sep}(k)$ ,  $\mathbf{CSep}(k)$  and  $\mathbf{CSA}(k)^{\oplus}$  are additive symmetric monoidal categories.

**Theorem 2.1.** [7, Corollary 2.13] There is an equivalence of categories

$$\{U(k)^{\oplus n} | n \in \mathbb{N}\} \simeq \mathbf{CSA}(k)^{\oplus} \times_{\mathbf{Sep}(k)} \mathbf{CSep}(k),$$

i.e.  $\{U(k)^{\oplus n} | n \in \mathbb{N}\}$  is a 2-pullback of categories with respect to the obvious inclusion morphisms.

For a central simple algebra  $A$  over  $k$ , denote by  $\mathrm{ind}(A)$  and  $\mathrm{deg}(A)$  the index (respectively degree) of  $A$ . Then by [2, Proposition 4.5.16],  $A$  admits a  $p$ -primary decomposition

$$A = \bigotimes_{i=1}^k A^{p_i},$$

where  $A^{p_i}$  is uniquely characterised by the property  $\mathrm{ind}(A^{p_i}) = p_i^{n_i}$  if

$$\mathrm{ind}(A) = p_1^{n_1} \cdots p_k^{n_k}$$

is the primary decomposition.

**Theorem 2.2.** [7, Theorem 2.19] Given central simple  $k$ -algebras  $A_1, \dots, A_n$  and  $B_1, \dots, B_m$ , the following two conditions are equivalent:

1. There is an isomorphism of noncommutative motives:

$$U(A_1) \oplus \dots \oplus U(A_n) \simeq U(B_1) \oplus \dots \oplus U(B_m).$$

2. The equality  $n = m$  holds, and for all  $1 \leq j \leq n$  and all  $p$

$$[B_j^p] = [A_{\sigma_p(j)}^p]$$

holds in  $\text{Br}(k)$ , for some permutations  $\sigma_p$  depending on  $p$ .

**Remark 2.3.** Though the isomorphism classes of objects in  $\text{CSA}(k)^\oplus$  are in some sense understood by Theorem 2.2, this is not true for  $\text{CSep}(k)$ . In fact, using the (additive) equivalence  $\text{CSep}(k) \simeq \text{Perm}(G)$ , where  $G = \text{Gal}(k_{\text{sep}}/k)$ , and  $\text{Perm}(G)$  is the category of permutation  $G$ -modules, interesting examples can be obtained from integral representation theory, see [7, Remark 2.5, 2.6].

### 3 Brauer-Severi varieties and full étale exceptional collections

Denote by  $BS(A)$  the Brauer-Severi variety associated to a central simple  $k$ -algebra  $A$ . We will say (see also [5]) that an object  $E \in D^b(BS(A))$  satisfying  $\text{Hom}(E, E[i]) = 0$  for all  $i \neq 0$  is

- semi-exceptional if  $\text{Hom}(E, E) = S$  is a semisimple  $k$ -algebra,
- étale exceptional if  $\text{Hom}(E, E) = L$  is an étale  $k$ -algebra.

It is well known [1] that  $BS(A)$  has a full semi-exceptional collection giving rise to a semi-orthogonal decomposition

$$D^b(BS(A)) = \langle D^b(k), D^b(A), \dots, D^b(A^{\otimes \deg(A)-1}) \rangle. \quad (3.1)$$

The following theorem now provides a positive answer to Conjecture 1.1.

**Theorem 3.1.** Non-split Severi-Brauer varieties do not admit full étale exceptional collections.

*Proof.* Suppose  $A$  is non-split and  $\deg(A) = d$ . Then if  $BS(A)$  has a full étale exceptional collection, we deduce from (3.1) and additivity of  $U(-)$  with respect to semi-orthogonal decompositions that there is an isomorphism

$$U(k) \oplus U(A) \oplus \dots \oplus U(A^{\otimes d-1}) \simeq U(D^b(BS(A))) \cong U(L_1) \oplus \dots \oplus U(L_d),$$

where the  $L_i$  are étale  $k$ -algebras. Using Theorem 2.1 and the universal property of fibre products, this isomorphism gives rise to an isomorphism

$$U(k) \oplus U(A) \oplus \cdots \oplus U(A^{\otimes d-1}) \simeq U(k)^{\oplus d}.$$

Now by Theorem 2.2, for all  $p : [A^p] = [k]$  in  $\mathrm{Br}(k)$ , so  $[A] = [k]$  or in other words  $A$  should split.  $\square$

**Remark 3.2.** This result formalizes (in this case) the intuition that for varieties defined over arbitrary fields, one should consider semi-exceptional collections instead of usual exceptional collections.

## References

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